Relating Vierbein formulation and Tensor formulation of Relativity

Here I took the material from *Nakahara* which should work fine. Captial indices denote non-coordinate indices and Greek indices denote the usual GR indices. Captial indices are raised or lowered with η_{IJ} and Greek indices are lowered or raised with $g_{\mu\nu}$. Vierbiens (one -forms and vector fields) are defined as follows:

$$e_I = e_I^\mu \frac{\partial}{\partial x^\mu}, \qquad e^I = e_\mu^I dx^\mu \tag{1}$$

DEFINITIONS:

$$\begin{aligned} g_{\mu\nu} &= e_\mu^I e_\nu^J \eta_{IJ} && (2) \\ \eta_{IJ} &= e_I^\mu e_J^\nu g_{\mu\nu} && (3) \\ e_\mu^I e_I^\nu &= \delta_\mu^\nu && (4) \\ e_\mu^I e_J^\mu &= \delta_J^I && (5) \end{aligned}$$

Relating coordinate and non-coordinate basis:

$$\text{For any vector field } V, \quad V = \underbrace{V^\mu e_\mu}_{\text{coordinate basis}} = \underbrace{V^I e_I}_{\text{non-coordinate basis}} = V^I e_I^\mu e_\mu \tag{6}$$

Think of it as writing the metric in this way:

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^{nu} = \eta_{IJ}\, e^I \otimes e^J \tag{7}$$

Now we show how to get the tensor formulation from the vierbien formulation of Einstein Equation. We need the following facts (can be proven):

$$R_\mu^I e_I^\nu = R_\mu^\nu \tag{8}$$

Hence it should be clear how to change Greek indices to Capitial indices and vice versa. Now we start with the Einstein Equation given in *Carlo Rovelli's* book (without the cosmological constant).

$$R_\mu^I - \frac{1}{2} R e_\mu^I = 0 \tag{9}$$

$$R_\mu^I e_I^\nu - \frac{1}{2} R e_\mu^I e_I^\nu = 0 \tag{10}$$

$$R_\mu^\nu - \frac{1}{2} R \delta_\mu^\nu = 0 \tag{11}$$

$$R_\mu^\nu g_{\nu\sigma} - \frac{1}{2} R \delta_\mu^\nu g_{\nu\sigma} = 0 \tag{12}$$

$$R_{\mu\sigma} - \frac{1}{2} R g_{\mu\sigma} = 0 \quad \text{(and we have it!)} \tag{13}$$

Hope this document is of any help to anybody, thanks.