These exercises are meant to deepen the understanding on spin networks and representations of su(2). Feel free to attack them in any order. Tedious exercises are marked with * while difficult ones are marked with **.

Exercise 1: Let $a, b, c \in \mathbb{R}_{\geq 0}$ show, if $|a-b| \leq c \leq a+b$ it follows that $|b-c| \leq a \leq b+c$. The consequence of this exercise is, that one needs to check the compatibility condition of trivalent vertices only for one of the spins.

Exercise 2: Let D_j , $2j \in \mathbb{N}$, denote the irreducible representations of SU(2) acting on vector spaces V_j. Recall, that any tensor representation of SU(2) can be uniquely decomposed into irreducible representations (the representation and the vector space the representation acts on are only distinguished if necessary):

$$V_j \bigotimes V_k = \bigoplus_{n=|j-k|}^{j+k} V_n \tag{1}$$

a) Show that the tensor representation of any three representations j,k,l (i.e. $V_j \bigotimes V_k \bigotimes V_l$ contains the singlet subspace (i.e. V_0) if and only if j+k+l is integer and the equation in ex.1 holds for j,k,l.

b) This time choose four representations j,k,l,m and show that the criterion for the existence of a singlet is j+k+l+m integer and:

$$|l-m| \leq j+k \leq l+m \qquad \text{or} \tag{2}$$

$$|l-m| \leq |j-k| \leq l+m \tag{3}$$

*c) In the spirit of ex.1) show that this criterion is invariant of the order of the indices.

*d) Try to generalize this result to more than four representations.

e) Due to the fact derived in c) we will now apply a natural order the representations in increasing order, i.e. $j \leq k \leq l \leq m$. See how the criterion b) simplifies and derive a formula for the dimension of the singlet space (i.e. how many copies of V_0 exist).

Exercise 3) Recall the representation theory of $\mathfrak{su}(2)$. Denote the three (hermitian) standard generators by L_1, L_2, L_3 with commutation relation $[L_i, L_j] = i\epsilon_{ijk}L_k$

a) Show, that $L^2 := L_1^2 + L_2^2 + L_3^2$ is a Casimir operator, i.e. commutes with every other operator. Define ladder operators $L_\pm := L_2 \mp L_1$. Compute L_+L_-, L^2 expressed in L_3, L_+ and L_-. Calculate the commutation relation for L_3 with L_+ and L_-.

*b) Use the fact, that L^2 and L_3 commute, so they can be mutually diagonalized, and that L^2 is a positive operator to find all irreducible representations of $\mathfrak{su}(2)$.

**c) Take two irreducible representation of $\mathfrak{su}(2)$, denote the generators $L_i^{(1)}$ and $L_i^{(2)}$ accordingly. Define the composite angular momentum operators as $L_i := L_i^{(1)} + L^{(2)}$. Perform the Clebsch-Gordan decomposition, i.e. show

$$|JM> = \sum_{m_1+m_2=M} |j_1 m_1 j_2 m_2 >< j_1 m_1 j_2 m_2 | JM > \tag{4}$$

with $J = j_1 + j_2$ and the scalar product denoting the Clebsch-Gordan coefficient. Compute the CG coefficient for simple examples.

d) Compute $< jmj(-m)|00 >$.

e) Show that for three representations i,j,k we have

$$|00> = \sum_{m_1 m_2 m_3} |im_1 jm_2 km_3 >< im_1 jm_2 | km_3 > \tag{5}$$

or something similar. (This formula justifies the the definition of 3-valent spin networks.)

Exercise 4 Assume A is pure gauge. Evaluate the spin networks show in figure 1.

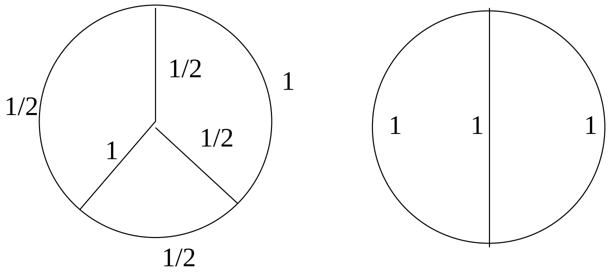

Figure 1: Spin networks

Exercise 5 Referring to page 20, figure 3 of Nicolai's paper: Use natural order of the label (ex 2e) and determine the number of possible values for k under the constraint that figure 3 is supposed to be a valid spin network. Compare this result with the dimension of the singlet subspace (2e)

Exercise 6

a) Verify that every SU(2) matrix in the standard representation can be written as

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \tag{6}$$

with $|a|^2 + |b|^2 = 1$

b) Define the Pauli matrices :

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7}$$

Show, that $\Sigma_k(t) := \exp(i\sigma_k t)$ is an element of SU(2) and compute $\Sigma_k(\frac{\pi}{2})$

**c) Look at figure 2. Assume the Wilson lines have already been computed and their values are given by $\Sigma_k(\frac{\pi}{2})$ as indicated by the circles in the diagram. Evaluate the spin networks.

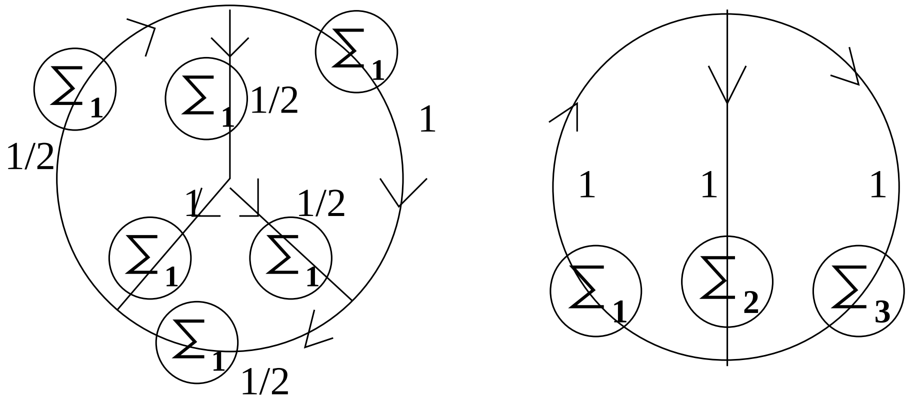

Figure 2: Spin networks with non trivial Wilson lines

Exercise 7 Let V be a vector space. The dual space V^* of V is defined as the space of all linear real valued functions. (Physicists might want to think of this space as a vector index lowered by a Kronecker delta metric)

a) Show that $Hom(V, W) \approx V^* \bigotimes W$.

b) Let there be a tensor group representations acting on $V^* \bigotimes W$. Show, that each singlet on the right side of ex. 7a corresponds exactly to an intertwining operator (element T of $Hom(V, W)$ which fulfills $g^{-1}T(gv) = T(v)$) on the left side.