The Ashtekar Hamiltonian for General Relativity

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We derive the Hamiltonian formulation of general relativity based on Ashtekar's variables [1]. This formulation and its variants are used for the canonical quantization of general relativity [2]. This presentation is based mainly on the book by John Baez [3] and the introductory paper by Domenico Giulini [4].

We first show how the connection can be considered as an independent degree of freedom together with the tetrad (Palatini formulation). The connection's six real degrees of freedom can then be replaced by three complex numbers; the self-dual part of the complex connection. The Lagrangian formulation in terms of the self-dual connection and the tetrad yield a relatively simple Hamiltonian formulation in which their spatial parts happen to be canonical conjugates.

1 The Palatini formulation

In the usual (Einstein-Hilbert) formulation of General Relativity, one starts with the metric $g_{\mu\nu}$ as the fundamental field; the only degree of freedom of the theory, beside matter. The Levi-Civita connexion ∇ is derived from $g_{\mu\nu}$; it is the unique torsion-free connexion compatible with the metric field. The equations of motion for the metric are given by variation of the Einstein-Hilbert action

$$S[g] = \int R[g]\Omega \tag{1}$$

where $R[g]$ is the curvature scalar derived from the Levi-Civita connexion and Ω the volume form.

Alternatively one can consider the tetrad field e^I_μ as the fundamental degree of freedom of the theory. Then the metric is computed as $g_{\mu\nu} = \eta_{IJ} e^I_\mu e^J_\nu$ and the connexion is derived as before from $g_{\mu\nu}$. The main difference with the initial approach is that the tetrad possesses new unphysical degrees of freedom since any Lorentz transformation on the indices I and J will yield the same metric.

In the Palatini formulation one starts instead with two fundamental degrees of freedom: a frame field and a connexion. It is then the variation of the action which yields the link between the connexion and the tetrad. We have to be precise in our definition of those variables. The connection will not be living on the tangent bundle but on a different non-natural vector bundle. Although all the equations will be the same, this point of view is needed in order to view the tetrad and the connexion's components as independent variables. For instance the variables representing the connexion will be invariant under variations of the tetrad. This would not be the case if those variables were seen as components with respect to the frame on the tangent bundle defined by the tetrad.

Let $\mathcal{M}$ be the four-dimensional manifold representing spacetime. We consider (at least locally) a frame field, or tetrad, e to be a vector bundle isomorphism between the tangent bundle $T\mathcal{M}$ and the trivial bundle $\mathcal{M} \times \mathbb{R}^4$ (a trivialization):

$$e : T\mathcal{M} \to \mathcal{M} \times \mathbb{R}^4$$

We interpret the fibre $\mathbb{R}^4$ at each point p of $\mathcal{M}$ as a local representation of $T_p\mathcal{M}$ for a free-falling observer. This means that if the map e is specified then we can compute the metric on $\mathcal{M}$. Indeed at each point p, the metric on $\mathcal{M}$ must be the image of the Lorentz metric η on $\mathbb{R}^4$:

$$g(v, w) = \eta(e(v), e(w))$$

for any two vectors v and w in $\mathbb{R}^4$. At each point p of $\mathcal{M}$, e is a linear map between two four-dimensional linear space. We can therefore write it in terms of its component e^I_μ and the above equation becomes

$$g_{\mu\nu} = \eta_{IJ} e^I_\mu e^J_\nu$$

We use Greek indices for components in $T_p\mathcal{M}$ and upper-case Latin letters for components in $\mathbb{R}^4$ (lower-case Latin letters will be reserved for space-like indices on $T_p\mathcal{M}$). If we write $\bar{e}^\mu_I$ for components of the inverse map we have

$\bar{e}^\mu_I e^I_\nu = \delta^\mu_\nu$. However if we move the Greek indices using the metric $g_{\mu\nu}$ and the Latin indices using η_{IJ} then

$$e^\mu_I e^I_\nu = g^{\mu\alpha}\eta_{KJ} e^K_\alpha e^I_\nu = g^{\mu\alpha} g_{\nu\alpha} = \delta^\mu_\nu$$

hence it is consistent to write $\bar{e}^\mu_I = e^\mu_I$.

We introduce another fundamental degree of freedom as a connexion D on $\mathcal{M} \times \mathbb{R}^4$. Let us write ξ_I for the vectors of the natural basis of $\mathbb{R}^4$. If v is a tangent vector in $T\mathcal{M}$ and s a vector in $\mathbb{R}^4$, the effect of D_v can be written as

$$D_v s = (v(s^J) + \omega^J_{\mu I} v^\mu s^I)\xi_J.$$

where $v(s^J)$ is the result of the action of the vector field v on the scalar field s^J on $\mathcal{M}$. The connexion is therefore fully specified by the object $\omega^J_{\mu I}$. We note that the connexion D does not have a “torsion” since this notion can be defined only for a directional derivative acting on tangent vectors. However we require that it preserves the Lorentz metric:

$$D_v \eta(s, s') = v\eta(s, s') = \eta(D_v s, s') + \eta(s, D_v s') \tag{2}$$

for any v of $T_p\mathcal{M}$ and any s, s' sections of $\mathcal{M} \times \mathbb{R}^4$. This implies that the matrices ω_μ belong to the Lie algebra of the Lorentz group $SO(1,3)$, hence

$$\omega^{JI}_\mu = -\omega^{IJ}_\mu$$

Via the isomorphism e, the connexion D defines a connexion ∇ on $T\mathcal{M}$:

$$\nabla_v w \doteq \bar{e}(D_v e(w)) \tag{3}$$

in components we get

$$\nabla_\mu \partial_\nu = e^\alpha_J(\partial_\mu e^J_\nu + \omega^J_{\mu I} e^I_\nu)\partial_\alpha \doteq \Gamma^\alpha_{\mu\nu}\partial_\alpha \tag{4}$$

with

$$\Gamma^\alpha_{\mu\nu} = e^\alpha_J(\partial_\mu e^J_\nu + \omega^J_{\mu I} e^I_\nu).$$

We can now compute the components of its torsion tensor. They are

$$\Gamma^\alpha_{[\mu\nu]} = e^\alpha_J(\partial_{[\mu} e^J_{\nu]} + \omega^J_{[\mu I} e^I_{\nu]}) = e^\alpha_J T^J_{\mu\nu}.$$

which defines the torsion two-form

$$T^J = de^J + \omega^I_J \wedge e^J = (De)^J$$

T codes for the torsion of the connexion ∇ and does not have to be zero. Therefore ∇ is not (yet) the Levi-Civita connexion. However it is compatible with the metric induced by the tetrad field e because D preserves the Lorentzian metric. Indeed, using equations (3) and (2) we have

$$\begin{aligned}\nabla_v g(u,w) &= v\eta(e(u),e(w)) \\ &= \eta(D_v e(u),e(w)) + \eta(e(u),D_v e(w)) \\ &= \eta(e(\nabla_v u),e(w)) + \eta(e(u),e(\nabla_v w)) \\ &= g(\nabla_v u,w) + g(u,\nabla_v w)\end{aligned}$$

(In fact every objects defined on the bundle $\mathcal{M}\times\mathbb{R}^4$ are mapped to an object with identical properties on $T\mathcal{M}$ through the bundle morphism e).

We see that the variable ω induces a connexion on $T\mathcal{M}$ which fails to be the Levi-Civita connexion associated with e only because it is not torsion-free: the object T^J does not vanish. It is now in fact the action that will enforce $T^J = 0$.

The action that we want is just the Einstein-Hilbert action (1) where the curvature scalar R of the Levi-Civita connection is replaced by the one induced by our connection ∇. The curvature of D is

$$F^{IJ}_{\mu\nu} = \omega^{IJ}_{[\mu,\nu]} + \omega^{I}_{[\mu K}\omega^{KJ}_{\nu]}$$

which is antisymmetric in both sets of indices. It can also be written

$$F^{IJ} = d\omega^{IJ} + \omega^{I}_{K}\wedge\omega^{KJ} \tag{5}$$

The bundle morphism e takes this curvature to that of ∇:

$$R^{\mu}{}_{\alpha\beta}{}^{\nu} = F^{IJ}_{\alpha\beta}e^{\mu}_{I}e^{\nu}_{J}$$

The Ricci tensor is then

$$R_{\alpha}{}^{\nu} = F^{IJ}_{\alpha\beta}e^{\beta}_{I}e^{\nu}_{J}$$

and the curvature scalar

$$R = R_{\alpha}{}^{\alpha} = e^{\alpha}_{I}e^{\beta}_{J}F^{IJ}_{\alpha\beta}$$

Hence the *Palatini action* is

$$S[e,\omega] = \int e^{\alpha}_{I}e^{\beta}_{J}F^{IJ}_{\alpha\beta}\Omega$$

We want to show that the variations of this action with respect to the connection variable ω is zero only when ∇ is torsion-free and that the variations with respect to the tetrad e yields Einstein's equations.

We first consider variations with respect to ω. We note that

$$\delta F^{IJ}_{\mu\nu} = \delta\omega^{IJ}_{[\mu,\nu]} + \omega^{I}_{[\mu K}\delta\omega^{KJ}_{\nu]} + \omega^{J}_{[\mu K}\delta\omega^{IK}_{\nu]}$$

hence

$$\delta S = \int e^{\mu}_{I} e^{\nu}_{J} (\delta\omega^{IJ}_{[\mu,\nu]} + 2\omega^{[I}_{[\mu K}\delta\omega^{KJ]}_{\nu]})\Omega$$

We consider the two terms separately. For the first one we have

$$\int e^{\mu}_{I} e^{\nu}_{J} \delta\omega^{IJ}_{[\mu,\nu]}\Omega = -\int \partial_\nu (e^{\mu}_{[I} e^{\nu}_{J]} e)\, \delta\omega^{IJ}_{\mu} \frac{1}{e}\Omega$$

where we used

$$\int \partial_\nu (e e^{\mu}_{[I} e^{\nu}_{J]} \delta\omega^{IJ}_{\mu}) \frac{1}{e}\Omega = 0$$

which follows from the fact that the integrand is a total divergence and that we do the variation on a volume on the boundary of which $\delta\omega^{IJ}_{\mu} = 0$.

The second term is

$$\begin{aligned} 2\int e^{\mu}_{I} e^{\nu}_{J} \omega^{[I}_{[\mu K}\delta\omega^{KJ]}_{\nu]}\Omega &= 2\int e^{\mu}_{[I} e^{\nu}_{J]} \omega^{I}_{\mu K}\delta\omega^{KJ}_{\nu}\Omega \\ &= -2\int e^{\mu}_{[K} e^{\nu}_{J]} \omega^{K}_{\nu I}\, \delta\omega^{IJ}_{\mu}\Omega \end{aligned}$$

Putting back both terms together we have

$$\begin{aligned} \delta S &= -\int \Big(\partial_\nu (e^{\mu}_{[I} e^{\nu}_{J]} e) + 2\omega^{K}_{\nu I}(e^{\mu}_{[K} e^{\nu}_{J]} e)\Big)\, \delta\omega^{IJ}_{\mu}\, \frac{1}{e}\Omega \\ &= -\int D_\nu (e^{\mu}_{[I} e^{\nu}_{J]} e)\, \delta\omega^{IJ}_{\mu}\, \frac{1}{e}\Omega \end{aligned}$$

Given the identity [5]

$$e e^{\mu}_{[I} e^{\nu}_{J]} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\epsilon_{IJKL} e^{K}_{\alpha} e^{L}_{\beta}$$

we get

$$\delta S = -\frac{1}{2}\int D_\nu (\epsilon^{\mu\nu\alpha\beta}\epsilon_{IJKL} e^{K}_{\alpha} e^{L}_{\beta})\, \delta\omega^{IJ}_{\mu}\, \frac{\Omega}{e}$$

Which gives that the covariant derivative of the tetrad must vanish [5]:

$$T^J = (De)^J = 0 \tag{6}$$

Hence the first equation of motion implies the vanishing of the torsion of the connection ∇ on $T\mathcal{M}$.

Next we consider the variations of the Palatini action with respect to e only. We have

$$\delta\Omega = -\frac{1}{2} g_{\alpha\beta}\delta g^{\alpha\beta}\Omega = -e^J_\alpha \delta e^\alpha_J \Omega$$

Then

$$\begin{aligned}
\delta S &= \int \left(\delta e^\alpha_I e^\beta_J F^{IJ}_{\alpha\beta} + e^\alpha_I \delta e^\beta_J F^{IJ}_{\alpha\beta} - R\, e^J_\alpha \delta e^\alpha_J\right)\Omega \\
&= 2\int \left(e^\alpha_I F^{IJ}_{\alpha\beta} - \frac{1}{2} R\, e^J_\beta\right)\delta e^\beta_J \Omega \\
&= 2\int \left(e^\alpha_I F^{IJ}_{\alpha\beta} - \frac{1}{2} R\, e^J_\beta\right) e^\gamma_J e^K_\gamma \delta e^\beta_K \Omega \\
&= 2\int \left(e^\alpha_I e^\gamma_J F^{IJ}_{\alpha\beta} - \frac{1}{2} R\, e^J_\beta e^\gamma_J\right) e^K_\gamma \delta e^\beta_K \Omega \\
&= 2\int \left(R^\alpha{}_{\alpha\beta}{}^\gamma - \frac{1}{2} R\, \delta^\gamma_\beta\right) e^K_\gamma \delta e^\beta_K \Omega \\
&= 2\int \left(R_{\beta\gamma} - \frac{1}{2} R\, g_{\gamma\beta}\right) e^{K\gamma} \delta e^\beta_K \Omega
\end{aligned}$$

which vanishes only when

$$R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta} = 0. \tag{7}$$

This is the vacuum Einstein equation provided that ∇ is torsion-free.

2 The self-dual formalism

On a four-dimensional Riemannian manifold, the Hodge star $*$ linearly sends two-forms to two-forms and satisfies $** = \mathbb{1}$. It therefore defines an involution on the space of two-forms with eigenvalues ± 1. This implies that every two-form T can be decomposed into a self-dual and an anti-self-dual part satisfying respectively $*T = T$ and $*T = -T$. They are the linear components of T in the eigenspace of $*$, respectively for eigenvalue 1 and -1. On a Lorentzian manifold however, the Hodge star satisfies $** = -\mathbb{1}$ and has the imaginary eigenvalues $\pm i$. Since now $*T = T$ as no non-trivial solution, self-duality is rather defined as $*T = iT$ and anti-self-duality as $*T = -iT$. This requires the usage of complex rather than real linear structures.

We can take the Palatini formalism and replace all real vector spaces by complex ones. For instance we replace the tangent bundle by the complexified tangent bundle $\mathbb{C}T\mathcal{M}$. It is defined from the real tangent bundle by allowing for all complex combination of tangent vectors at each point. We introduced the complex frame field as a map

$$e : \mathbb{C}T\mathcal{M} \to \mathcal{M} \times \mathbb{C}^4$$

Similarly we replace the connection D by a connection on $\mathcal{M} \times \mathbb{C}^4$ that we represent by the same components ω_μ^{IJ}. Then all the equations of the Palatini formulation are still valid, now referring to complex objects.

We do not consider the Hodge star defined on the natural cotangent bundle but rather its equivalent on $\mathcal{M} \times \mathbb{C}^4$. Instead of two-tensors, with two antisymmetric spacetimes indices, we consider objects with two antisymmetric Lorentz (upper-case Latin) indices. Those object can be considered as taking value in the Lie algebra of the (complexified) Lorentz group $so(1,3)_\mathbb{C}$. This applies for instance to ω_μ^{IJ} or its curvature $F_{\mu\nu}^{IJ}$. By analogy with the Hodge star on two-forms we then define the operator $\star$ as on $so(1,3)$ as:

$$\star T^{IJ} = \frac{1}{2}\epsilon^{IJ}{}_{KL}T^{KL}$$

which satisfies $\star\star = -\mathbb{1}$. We say that T is self-dual (anti-self-dual) if it belongs to the eigenspace of $\star$ of eigenvalue i ($-i$):

$$\star T = \pm iT.$$

The projectors on the two eigenspaces are

$$P^\pm = \frac{1}{2}(\mathbb{1} \mp i\star).$$

This decomposition into eigenspaces is not only compatible with the linear structure of $so(1,3)_\mathbb{C}$ but also with its full Lie algebra structure. The Lie bracket is

$$[F, G]^{IJ} = F^I{}_K G^{KJ} - G^I{}_K F^{KJ}.$$

We show that the Lie bracket preserves the self-dual and anti-self-dual subspaces. We have

$$\begin{aligned}\star[F,\star G]^{IJ} &= \frac{1}{4}\epsilon^{IJ}{}_{KL}F^{K}{}_{M}\,\epsilon^{ML}{}_{PQ}G^{PQ}\\ &\quad-\frac{1}{4}\epsilon^{IJ}{}_{KL}\epsilon^{K}{}_{MPQ}G^{PQ}F^{ML}\\ &= \frac{1}{4}(\epsilon^{IJ}{}_{KL}\epsilon^{ML}{}_{PQ}-\epsilon^{IJLM}\epsilon_{LKPQ})F^{K}{}_{M}\,G^{PQ}\\ &= -\frac{1}{2}\epsilon^{IJKL}\epsilon_{MLPQ}F^{M}{}_{K}\,G^{PQ}\end{aligned}$$

Because of the antisymmetry of F we must have $K \neq M$, which implies either $K = P$ or $K = Q$. Considering both cases separately and using the identity

$$\epsilon_{IJKL}\epsilon^{IJMN} = 2(\delta_{K}{}^{M}\delta_{L}{}^{N} - \delta_{K}{}^{N}\delta_{L}{}^{M})$$

we obtain

$$\star[F,\star G] = -[F,G]$$

using also $\star\star = -\mathbb{1}$, it follows that $\star[F,G] = [\star F,G] = [F,\star G]$. Since those last identities are true for the identity operator too, we have

$$P^{\pm}[F,G] = [P^{\pm}F,G] = [F,P^{\pm}G]. \tag{8}$$

This proves that the two subspaces of $so(1,3)_{\mathbb{C}}$ defined by the projectors $P^{\pm}$ are ideals of the Lie algebra. Therefore we can write

$$so(1,3)_{\mathbb{C}} = so(1,3)^{+}_{\mathbb{C}} \oplus so(1,3)^{-}_{\mathbb{C}} \tag{9}$$

where $so(1,3)^{\pm}_{\mathbb{C}}$ contains only the self-dual (anti-self-dual) elements of $so(1,3)_{\mathbb{C}}$.

Let

$${}^{+}\omega \doteq P^{+}\omega = \frac{1}{2}(\omega - i\star\omega)$$

be the self-dual part of the complex connection ω. The above calculations guarantee that this stays self-dual under $so(1,3)^{+}_{\mathbb{C}}$ gauge transformations. Its curvature is

$${}^{+}F^{IJ}{}_{\mu\nu} = {}^{+}\omega^{IJ}{}_{[\mu,\nu]} + {}^{+}\omega^{I}{}_{[\mu K}\,{}^{+}\omega^{KJ}{}_{\nu]}$$

which can also be written

$$\begin{aligned}{}^{+}F_{\mu\nu} &= P^{+}\omega_{[\mu,\nu]} + \frac{1}{2}[P^{+}\omega_{\mu}, P^{+}\omega_{\nu}]\\ &= P^{+}\omega_{[\mu,\nu]} + P^{+}\frac{1}{2}[\omega_{\mu},\omega_{\nu}]\\ &= P^{+}F_{\mu\nu}\end{aligned}$$

This means that the self-dual component of the curvature of ω is the curvature of the self-dual part of ω. Therefore the complex Palatini action can be written in terms of a self-dual and an anti-self-dual part which depend respectively only on the self-dual and anti-self-dual connections:

$$S^{+}[e, {}^{\pm}\omega] = \int e_I^{\alpha} e_J^{\beta}\, {}^{\pm}F^{IJ}{}_{\alpha\beta}\,\Omega$$

such that

$$S[e, \omega] = S^{+}[e, {}^{+}\omega] + S^{-}[e, {}^{-}\omega]$$

We have defined the self-dual complex connection ${}^{+}\omega$ from a real connection ω. Inversely it is clear that a self-dual connection ${}^{+}\omega$ uniquely define the real connection to which it derives through

$$\omega = \frac{1}{2}({}^{+}\omega + {}^{+}\bar{\omega})$$

where the bar denotes complex conjugation. Therefore the knowledge of ${}^{+}\omega$ is equivalent to the knowledge of the real ω. We can therefore write

$$S^{+}[e, {}^{+}\omega] = S[e, \omega] - \frac{1}{2} i T[e, \omega]$$

where

$$T[e, \omega] \doteq \int e_I^{\alpha} e_J^{\beta} (\star F)^{IJ}{}_{\alpha\beta}\Omega$$

and both components S and T are real. Therefore, for any type of variations we have that $\delta S^{+} = \delta S - \frac{1}{2} i \delta T = 0$ if and only if the real part and the imaginary part are zero: $\delta S[e, \omega] = 0$ and $\delta T[e, \omega] = 0$. This implies that the equations of motion derived from S are also equations of motions for S^{+}. However one may fear that the second condition on T adds additional equations of motions which therefore reduce the space of solutions. We show that this is not the case. Variations of S^{+} can be done in almost the same way as the original Palatini action. Variations with respect to e yield

$$e_I^{\alpha}\, {}^{+}F^{IJ}_{\alpha\beta} = 0.$$

If the torsion defined on ω already vanishes, which follows from extremization of S, then this equation reduces to

$$\epsilon^{IJ}{}_{KL} F^{KL}_{\alpha\beta} e_I^{\alpha} = 0$$

which is trivially satisfied. This can be seen by multiplying the free Latin index with a tetrad and introducing two identities:

$$\begin{aligned} & 0 = \epsilon^{IJ}{}_{KL} F^{KL}_{\alpha\beta} e^{\alpha}_{I} \\ \Leftrightarrow\ & 0 = \epsilon^{IJ}{}_{MN} e^{\gamma}_{J} e^{M}_{\mu} e^{\mu}_{K} e^{N}_{\nu} e^{\nu}_{L} F^{KL}_{\alpha\beta} e^{\alpha}_{I} \\ \Leftrightarrow\ & 0 = \epsilon^{IJ}{}_{MN} e^{\gamma}_{J} e^{M}_{\mu} e^{\mu}_{K} e^{N}_{\nu} e^{\nu}_{L} F^{KL}_{\alpha\beta} e^{\alpha}_{I} \\ \Leftrightarrow\ & 0 = \epsilon^{\alpha\gamma}{}_{\mu\nu} e^{\mu}_{K} e^{\nu}_{L} F^{KL}_{\alpha\beta} \end{aligned}$$

which is always true given that the object in front of ϵ is symmetric under the joint exchange of α, β and μ, ν which can be undone by an antisymmetric exchange of indices on ϵ. Therefore variations of T with respect to the tetrad e does not bring any additional equation. Also, one can show that variations of T with respect to ω also trivially vanishes if the torsion is zero [6]. In fact variations of the self-dual action with respect to the self-dual connection imply that the connection must be the self-dual part of the Levi-Civita connection [4]. This shows that the self-dual action $S^{+}[e, {}^{+}\omega]$ suffices to define general relativity.

3 The Ashtekar formulation

The self-dual connection components ${}^{+}\omega^{IJ}$ are not convenient variables because they have constraints (antisymmetry and self-duality). Instead we consider the three complex variables

$$A^{a}_{\mu} \doteq 2\,{}^{+}\omega^{0a}_{\mu}$$

where the lowercase Latin indices run from 1 to 3 and is raised and lowered with the Kronecker delta δ_{ab}. Since ${}^{+}\omega$ is self-dual, the three complex numbers A^{a} code for all of its independent components. Indeed self-duality implies

$${}^{+}\omega^{0i} = -i\frac{1}{2}\epsilon^{0i}{}_{jk}\,{}^{+}\omega^{jk}$$

and

$${}^{+}\omega^{jk} = -i\epsilon^{jk}{}_{0i}\,{}^{+}\omega^{0i}$$

which means that in particular ${}^{+}\omega^{23} = i\,{}^{+}\omega^{01}$, ${}^{+}\omega^{13} = i\,{}^{+}\omega^{02}$ and ${}^{+}\omega^{12} = i\,{}^{+}\omega^{03}$. Those relations allow to easily compute the Lie bracket in the new variables:

$$[A, B]^{c} = 2i\epsilon^{c}{}_{ab}A^{a}B^{b}.$$

We note that this is the Lie bracket of the algebra $sl(2,\mathbb{C})$ in the basis given by the three Pauli matrices; with $A = A^a\sigma_a \in sl(2,\mathbb{C})$. This shows that the algebras $so(1,3)^{\pm}_{\mathbb{C}}$ introduced in (9) are isomorphic to $sl(2,\mathbb{C})$. Also, if the components A^a are purely imaginary then A is in a subalgebra isomorphic to $su(2)$, or $so(3)$. In that sense we can think of A as living in the complexification of $so(3)$.

The curvature of A^a is

$$F^a_{\mu\nu} = A^a_{[\mu,\nu]} + \frac{1}{2}[A_\mu, A_\nu]^a.$$

The spatial components A^a_i are part of Ashtekar's variables: they will serve as the canonical position for the Hamiltonian formalism.

3.1 Hamiltonian formalism

We now want to decompose spacetime into space and time. Following the notation in Ref. [2], we will write $i, j, k, l, \dots$ for the spacelike tangent vector components which run from 1 to 3, and $a, b, c, \dots$ for the Lorentz components running from 1 to 3.

We consider an arbitrary decomposition of the spacetime manifold into space and time $\mathcal{M} = \Sigma \times \mathbb{R}$. This suggests an appropriate coordinate system such that ∂_0 generates translations in this global time and the flow generated by ∂_i let the global spacelike slices Σ invariant. In addition we choose a tetrad e^μ_I such that the spacetime vectors e_a are everywhere tangent to the spacelike slices, which make e_0 orthogonal to Σ. This means that the components e^i_a define a triad on the spacetime slices. The vector field ∂_0 is generally not orthogonal to Σ. It's failure to do so is coded in the lapse function n and the shift N^a defined by

$$\partial_0 = ne_0 + N^a e_a.$$

Since also $e_a = e^i_a \partial_i$ which can be inverted to yield $\partial_i = e^a_i e^a$ we have the relation between ∂_μ an e_μ. Inverting this relation show that

$$e^0_0 = \frac{1}{n} \qquad e^i_0 = -\frac{N^i}{n} \qquad e^0_a = 0.$$

Therefore we can interpret n and N^i as new variables replacing the components e^μ_0. The last equation however amount to a partial fixing of the Lorentz gauge; it accounts for the fact that we consider only tetrads compatible with a given spacelike slicing.

From those equations, the dual triad is simply

$$e_i^a = g_{ij}\eta^{ab}e_b^j = h_{ij}\delta^{ab}e_b^j$$

where we have introduced the three-metric

$$h_{ij} = g_{ij} = \delta_{ab}e_i^a e_j^b$$

Given $e_a^0 = 0$ it is easy to compute

$$\det(e_I^\mu) = \frac{1}{n}\det(e_a^i)$$

which implies that $n\sqrt{h} = \sqrt{g}$. Therefore the volume form is

$$\Omega = \sqrt{h}\, n\, dt \wedge dx^1 \wedge dx^2 \wedge dx^3$$

We now define densitized version of the tetrad:

$$E_I^\mu \doteq \sqrt{h}\, e_I^\mu$$

and instead of E_0^0 we use the lapse

$$N \doteq \frac{1}{\sqrt{h}}\, n.$$

Plugging this into the expression for the self-dual action we directly obtain

$$S^+ = \int (E_a^i E_b^j\, {}^+F_{ij}^{ab} N + 2E_b^i\, {}^+F_{0i}^{0b} - 2N^i E_b^j\, {}^+F_{ij}^{0b})\, dt d^3x$$

where we have abbreviated the form $dt \wedge dx^1 \wedge dx^2 \wedge dx^3$ as dtd^3x.

We can now write the components of the self-dual curvature in terms of A_μ^a. We get

$${}^+F_{ij}^{0b} = \frac{1}{2}A_{[i,j]}^b + \frac{1}{4}[A_i, A_j]^b = \frac{1}{2}F_{ij}^b$$

where F^c is the curvature of A. Also

$${}^+F_{ij}^{ab} = -i\frac{1}{2}\,\epsilon^{ab}{}_c F_{ij}^c$$

And finally

$$
\begin{aligned}
{}^{+}F^{0b}_{0i} &= \frac{1}{2}\left(A^b_{[0,i]} + \frac{1}{2}[A_0, A_i]^b\right) \\
&= -\frac{1}{4}\left(\partial_0 A^b_i - D_i A^b_0\right)
\end{aligned}
$$

where we introduced the covariant spatial derivative (on Σ) defined by A^a_j,

$$
D_i x^b = \partial_i x^b - [A_i, x]^b \tag{10}
$$

With those results we can write the action in terms of the new variables:

$$
\begin{aligned}
S^+ = -\frac{1}{2}\int \Big[& (\partial_0 A^b_j) E^j_b \\
& - \left(A^b_0 D_j E^j_b - 2N^j E^i_b F^b_{ij} - i\epsilon^{ab}{}_c N E^i_a E^j_b F^c_{ij}\right)\Big]\, dt d^3x
\end{aligned}
$$

where the term $E^j_b D_j A^b_0$ has been integrated by part to yield $-A^b_0 D_j E^j_b$.

This action is now a function of A^a_i, and E^i_a as well as A^a_0, N^i and N which code for the non-spatial parts of the original tetrad and connection fields. Those are the variables introduced by Ashtekar [1]. We can choose A^a_i to be the canonical configuration variables since it uniquely defines a three-metric on the spatial surfaces. Then we see from the action that the corresponding momentum variables, defined as the result of the variations of the Lagrangian density as function of the time derivatives of A^a_i, is just E^i_a. Therefore A^a_i and E^i_a are the two conjugated canonical variables.

We already know that variations of the action with respect to all the variables will vanish exactly when they satisfy Einstein's equations. Variation of the action with respect to the canonical variables A^a_i and E^i_a will yield Hamilton's equations. But the action also has to vanish for variations with respect to A^a_0, N^j and N. This implies that the spatial fields A^a_i and E^i_a must satisfy the following equations on Σ:

$$
D_j E^j_b = 0 \tag{11}
$$

$$
E^i_b F^b_{ij} = 0 \tag{12}
$$

$$
\epsilon^{ab}{}_c E^i_a E^j_b F^c_{ij} = 0 \tag{13}
$$

Since they involve no time derivative, those equations are constraints. The objects on the left hand side of those equations, when integrated over space against a test function, are observables which generate canonical transformations. Since they must always vanish those observable are in particular conserved. Therefore, by Noether's theorem, they must generate symmetries of the theory. A way of understanding the meaning of those equations is to look at the symmetries that the corresponding observables generate. In fact the reason why those observables vanish on all physical configurations is that they generate gauge transformations [7].

In the tetrad formalism of general relativity there are two type of Gauge transformations; the diffeomorphisms and the local Lorentz transformations of the tetrad. When we separated spacetime into space and time we already fixed part of the Lorentz gauge by requiring three components of the tetrad to be tangent to the spacelike slices. Therefore we are left with the spatial rotations which are generated by the constraints (11). The two other constraints are related to diffeomorphisms. The constraints (12) generate all the spatial diffeomorphisms on Σ, and the scalar constraints (13) generate the timelike diffeomorphisms.

The Hamiltonian density is defined as $\mathcal{H} = (\partial_0 A_i^a) E_a^i - \mathcal{L}$ where $\mathcal{L}$ is Lagrangian density. This is

$$\mathcal{H}[A,E] = A_0^b D_j E_b^j - 2N^i E_b^i F_{ij}^b - i\epsilon^{ab}{}_c N E_a^i E_b^j F_{ij}^c$$

which is already a function of the right variables (A and E; there is no time derivative of A). We see that once the constraints (11), (12) and (13) are solved, the Hamiltonian identically vanishes. This is to be expected since the Hamiltonian generates time translations which are just a diffeomorphism; a gauge transformation already taken into account by the constraints [2].

Therefore Hamiltonian general relativity is determined only by constraints which define the physical phase space. This is actually not so different from any canonical theory; choosing a point of phase space uniquely determine the state of the system *at all time*. It does usually not matter whether one choose to interpret phase space as the space of states at a given time or as the space of *histories*; i.e. the space of solutions of the kinematical and dynamical equations. However in general relativity only the later interpretation is valid because any notion of time depends on the metric and therefore on the state itself [2] (unless one consider particular boundary conditions which allow for the definition of an external time, a possibility that we have neglected here by systematically removing boundary terms).

3.2 The reality condition

We did not yet address the problem that all the quantities considered are complex. What we have described is the canonical form of *complex* general relativity. In order to obtain real general relativity one has to consider only solutions for which the triad E^i_a is real. However this is not sufficient; one also has to make sure that the extrinsic curvature is real. The extrinsic curvature K_{ab} is defined by

$$\nabla_{e_a} e_b = {}^3\nabla_{e_a} e_b + K_{ab} e_0$$

where ${}^3\nabla$ is the Levi-Civita connection induced by the three metric h_{ij}. Since $\nabla_{e_a} e_b = e^\nu_a \omega^I_{\nu b} e_I$, and $e^0_a = 0$, expanding this equation yields

$$e^i_a \omega^c_{ib} e_c + e^i_a \omega^0_{ib} e_0 = e^i_a \omega^c_{ib} e_c + K_{ab} e_0$$

hence

$$K_{ab} = e^i_a \omega^0_{ib}$$

Plugging this into the definition of A^a_i in terms of $\omega^I_{\nu J}$ yields

$$\begin{aligned} A^a_i &= -i\frac{1}{2}\epsilon^{0a}{}_{bc}(\omega^{bc}_i - i\epsilon^{bc}{}_{0d}\omega^{0d}_i) \\ &= -i\frac{1}{2}\epsilon^{0a}{}_{bc}(\omega^{bc}_i - i\epsilon^{bc}{}_{0d} e^e_i K_e{}^d) \\ &= -i\frac{1}{2}\epsilon^{0a}{}_{bc}\omega^{bc}_i + e^d_i K_d{}^a \end{aligned}$$

If we make the natural definitions

$$\Gamma^a_i[E] \doteq -\frac{1}{2}\epsilon^{0a}{}_{bc}\omega^{bc}_i \qquad K^a_i \doteq e^c_i K_c{}^a$$

were the symbol $\Gamma[E]$ reminds us that we are working with the Levi-Civita connection defined by the triad E^a_i, then

$$A^a_i = K^a_i + i\Gamma^a_i[E]$$

which gives the relation between the new variable A^a_i and geometrical quantities. Since ω^{bc}_i and therefore Γ^a_i is real, the reality of K^a_i is given by the constraint

$$A^a_i - \bar{A}^a_i = 2i\Gamma^a_i[E] \tag{14}$$

between the canonical variables.

Finally we note that this implies that this formulation is not properly speaking a *canonical* formulation of *real* general relativity since this last equation is not part of the canonical formalism [8].

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